

MEMORANDUM

RM-3984-PR

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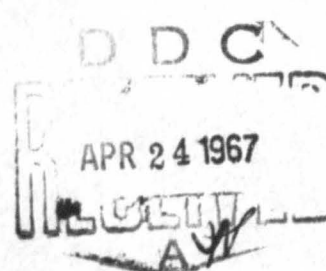
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INCIDENCE MATRICES WITH  
THE CONSECUTIVE 1's PROPERTY

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PREPARED FOR:

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## **PREFACE**

This Memorandum presents some mathematical results pertaining to the study of interval graphs and their application to a problem concerning the structure of genes. The mathematical statement of the particular genetic problem considered here is: Given a large number of mutant genes, together with intersection data on pairs of mutants, then the problem is to decide whether this information is compatible with a linear model of the gene. This means that one must determine whether the graph of the intersection data is an interval graph, and this can be done by considering a certain incidence matrix associated with the graph.

The research presented in this Memorandum is an example of the basic supporting studies in mathematics conducted by RAND mathematicians.

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### SUMMARY

✓ Let  $A = (a_{ij})$  be an  $m$ -by- $n$  matrix whose entries  $a_{ij}$  are all either 0 or 1. For certain applications it is of interest to know whether or not there is an  $m$ -by- $m$  permutation matrix  $P$  such that the 1's in each column of  $PA$  occur in consecutive positions. In this note certain results having relevance to this problem are stated. Proofs of these results together with computationally effective algorithms for deciding the question are to be published elsewhere.

The problem formulated above is directly related to that of determining whether a given finite, undirected graph is an interval graph. Although necessary and sufficient conditions are known for the solution of this latter problem, the approach used here is quite different from those used heretofore and seems to lead to highly efficient algorithms not only for resolving the question, but also for producing a representative set of intervals in the affirmative case.

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## 1. INTRODUCTION

Let  $A = (a_{ij})$  be an  $m$ -by- $n$  matrix whose entries  $a_{ij}$  are all either 0 or 1. For certain applications, one of which will be discussed below, it is of interest to know whether there is an  $m$ -by- $m$  permutation matrix  $P$  such that the 1's in each column of  $PA$  occur in consecutive positions. In this note we state certain results that have relevance for this problem. Proofs of these, together with an efficient computational method for deciding the question in any given case, will be published elsewhere.

The problem posed above includes that of determining whether a given finite, undirected graph is an interval graph. The study of interval graphs [2, 3, 4, 5] was stimulated in part by an application concerning the fine structure of genes. A basic genetic problem, discussed in [1], is to decide whether or not the sub-elements of genes are linked together in a linear order. A way of approaching this problem is also described in [1]. Briefly, it is as follows. For certain microorganisms, there are a standard form and mutants, the latter arising from the standard form by alteration of some connected portion of the genetic structure. Experiments can be devised for determining whether or not the blemished parts of two mutant genes intersect or not. Thus the mathematical

problem becomes: Given a large number of mutants together with intersection data on pairs of mutants, to decide whether this information is compatible with a linear model of the gene. If one represents the intersection data by a graph (two mutant genes, i.e., vertices, being joined by an edge if their blemished portions intersect), the problem is to decide whether this graph is an interval graph.

## 2. A BASIC THEOREM

We say that a  $(0, 1)$ -matrix  $A$  has the consecutive 1's property (for columns) if there is a permutation matrix  $P$  such that the 1's in each column of  $PA$  occur consecutively. The first question that naturally arises is how much information about  $A$  is needed to decide whether it has the property or not. Do we need to know  $A$  itself, or will something less suffice? Theorem 2.1 below provides a partial answer to this question; it shows that a knowledge of the matrix  $A^T A$  is enough. Here  $A^T$  denotes the transpose of  $A$ .

Theorem 2.1. Let  $A$  and  $B$  be  $(0, 1)$ -matrices satisfying

$$(2.1) \quad A^T A = B^T B .$$

Then either both  $A$  and  $B$  have the consecutive 1's property or neither does. Moreover, if  $A$  and  $B$  have the same number of rows and  $A$  has the property, then there is a permutation  $P$  such that  $B = PA$ .

The first part of Theorem 2.1 follows easily from the second. The second assertion can be proved by induction on the number of columns of  $A$ .

In view of Theorem 2.1, it would be interesting to know conditions on  $A^T A$  in order that  $A$  have the consecutive 1's property. Later on we shall state a theorem which reduces this question to the consideration of  $(0, 1)$ -matrices having connected "overlap graphs." For such matrices, there is a simple construction for testing the property, but we do not know explicit necessary and sufficient conditions.

### 3. THE OVERLAP GRAPH AND COMPONENT GRAPH

Let  $a$  and  $b$  be  $(0, 1)$ -vectors having  $m$  components. Their inner product  $a \cdot b$  satisfies

$$(3.1) \quad 0 \leq a \cdot b \leq \min(a \cdot a, b \cdot b) .$$

If strict inequality holds throughout (3.1), we say that  $a$  and  $b$  overlap. We also say that  $a$  contains  $b$  if

$$(3.2) \quad a \cdot b = b \cdot b .$$

Now let  $A$  be an  $m$ -by- $n$   $(0, 1)$ -matrix having column vectors  $a_j$ ,  $j = 1, 2, \dots, n$ . It is convenient, and presents no loss of generality in studying the consecutive 1's property, to assume that  $a_j \neq 0$ ,  $j = 1, 2, \dots, n$ , and that  $a_i \neq a_j$  for  $i \neq j$ . We refer to such an  $A$  as proper.

There are various graphs one can associate with a  $(0, 1)$ -matrix  $A$  that are meaningful insofar as the consecutive 1's property is concerned. We describe two such, one being an undirected graph, the other a directed graph. The first of these is obtained from  $A$  by taking vertices  $x_1, x_2, \dots, x_n$  corresponding to the columns  $a_1, a_2, \dots, a_n$  of  $A$  and putting in undirected edges  $(x_i, x_j)$  corresponding to overlapping column vectors  $a_i$  and  $a_j$ . We call this the overlap graph of  $A$  and denote it by  $\mathcal{H} = \mathcal{H}(A)$ . The overlap graph of  $A$  splits up into connected components  $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_p$ , and this decomposition yields a corresponding partition of  $A$  into  $m$ -rowed submatrices  $A_1, A_2, \dots, A_p$ . We now form a second (directed) graph by taking vertices  $X_1, X_2, \dots, X_p$  corresponding to these submatrices and putting in an edge  $[X_i, X_j]$  directed from  $X_i$  to  $X_j$  if there is a column vector  $a$  of  $A_i$  and a column vector  $b$  of  $A_j$  such that  $a$  contains  $b$ . We call this directed graph the component graph of  $A$  and denote it by  $\mathcal{D} = \mathcal{D}(A)$ .

The following theorem may be established in a straightforward manner.

Theorem 3.1. The component graph  $\mathcal{D}(A)$  of a proper  $(0, 1)$ -matrix  $A$  is acyclic and transitive.

That is,  $\mathcal{D}(A)$  contains no directed cycles, and if  $[X, Y]$  and  $[Y, Z]$  are edges, then  $[X, Z]$  is an edge. Thus  $\mathcal{D}(A)$  is the graph of a partial ordering. This partial ordering of components of  $\mathcal{H}(A)$  is special in the sense that an element can have at most one immediate predecessor.



Thus if we omit from  $\mathcal{D}$  every edge whose existence is implied by transitivity, the resulting graph is simply a collection of rooted trees.

The structure of the component graph  $\mathcal{D}(A)$  is useful in establishing the decomposition theorem of the next section.

#### 4. A DECOMPOSITION THEOREM

For an arbitrary  $(0, 1)$ -matrix  $A$ , we can rearrange columns and write

$$(4.1) \quad A = (A_1, A_2, \dots, A_p),$$

where each submatrix  $A_k$ ,  $k = 1, 2, \dots, p$ , corresponds to a component of the overlap graph  $\mathcal{G}(A)$ . We term (4.1) an overlap decomposition of  $A$ , and refer to the submatrices  $A_k$  as components of  $A$ . If  $A$  has just one component, we say that  $A$  is connected.

Theorem 4.1. A  $(0, 1)$ -matrix  $A$  has the consecutive 1's property if and only if each of its components has the property.

Necessity in Theorem 4.1 is of course trivial. Sufficiency can be established by induction on the number of components of a proper  $A$ . The induction step proceeds by deleting a component of  $A$  which corresponds to a minimal element in the partial ordering given by  $\mathcal{D}(A)$ .

Theorem 4.1 effectively solves the problem posed in Section 1, since one can describe a very simple and efficient

procedure for testing whether or not a connected matrix has the consecutive 1's property. Moreover, having arranged each individual component of a disconnected A so that its 1's appear consecutively in each column, the proof of Theorem 4.1 indicates how to fit these components together so as to yield a permuted form of A which has consecutive 1's in each column. The entire process is computationally efficient, requiring no more than  $O(n^2)$  steps if A has n columns.

## 5. APPLICATION TO INTERVAL GRAPHS

A graph  $\mathcal{G}$  (finite, undirected, without multiple edges or loops) is an interval graph provided  $\mathcal{G}$  can be represented as the intersection graph of a set of intervals on the real line. The theorems and methods described in preceding sections can be applied to the problem of determining when a graph  $\mathcal{G}$  is an interval graph by considering a certain incidence matrix which specifies  $\mathcal{G}$ . We term this incidence matrix the dominant-clique-vs.-vertex matrix, and define it as follows. First of all, a clique in  $\mathcal{G}$  is a set of vertices, every two of which are joined by an edge. We may partially order the set of all cliques of  $\mathcal{G}$  by inclusion. The maximal elements in this ordering will be termed dominant cliques. Since two vertices of  $\mathcal{G}$  are joined by an edge if and only if they belong to some dominant clique, the dominant-clique-vs.-vertex incidence matrix characterizes  $\mathcal{G}$ .

Theorem 5.1. A graph  $\mathcal{G}$  is an interval graph if and only if the dominant-clique-vs.-vertex incidence matrix of  $\mathcal{G}$  has the consecutive 1's property.

We also note that an interval graph is necessarily a rigid-circuit graph [2], and that one can describe a simple method to test for the rigid-circuit property. (A graph is a rigid-circuit graph if every circuit with more than three vertices has a chord. The test is based on the known fact that such a graph always contains simplicial vertices, a simplicial vertex being one whose neighboring vertices are a clique [2], [3].) If the test succeeds, the method automatically generates all dominant cliques. Thus to discover if  $G$  is an interval graph, one can first apply an easy test for the rigid-circuit property, and then test the resulting dominant-clique-vs.-vertex incidence matrix for the consecutive 1's property.

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